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Non-Douglas–Kazakov phase transition of two-dimensional generalized Yang–Mills theories

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Abstract. In two-dimensional Yang–Mills and generalized Yang–Mills theories for large gauge groups, there is a dominant representation determining the thermodynamic limit of the system. This representation is characterized by a density, the value of which should everywhere be between zero and one. This density itself is determined by means of a saddle-point analysis. For some values of the parameter space, this density exceeds one in some places. So one should modify it to obtain an acceptable density. This leads to the well-known Douglas–Kazakov phase transition. In generalized Yang–Mills theories, there are also regions in the parameter space where somewhere this density becomes negative. Here too, one should modify the density so that it remains nonnegative. This leads to another phase transition, different from the Douglas–Kazakov one. Here the general structure of this phase transition is studied, and it is shown that the order of this transition is typically three. Using carefully-chosen parameters, however, it is possible to construct models with the order of the phase transition not equal to three. A class of these non-typical models is also studied.

1 Introduction

The two-dimensional Yang–Mills theory (YM_2) is a laboratory for testing ideas and concepts of the Yang–Mills model in the real four-dimensional world. The string picture of YM_2 is also interesting in itself, as an example of a nonperturbative analysis of a quantum field theory [1–7].

The starting point to establish the correspondence between YM_2 and string theory is the study of the large-N limit of YM_2. For example, as it is shown in [3–5], a gauge theory based on $SU(N)$ is split at large N into two copies of a chiral theory, which encapsulate the geometry of the string maps. The chiral theory associated with the Yang–Mills theory on a two-manifold $\mathcal{M}$ is a summation over maps from a two-dimensional world sheet (of arbitrary genus) to the manifold $\mathcal{M}$. This leads to a $1/N$ expansion for the partition function and observables that is convergent for all of the values of area $\times$ coupling constant on the target space $\mathcal{M}$, if the genus is one or greater.

The large-N limit of the $U(N)$ YM_2 on a sphere has been studied in [8]. There the dominant (or classical) representation has been found, and it has been shown that the free energy of the $U(N)$ YM_2 on a sphere with surface area $A < A_c = \pi^2$ has a logarithmic behavior. In [9], the free energy has been calculated for areas $A > \pi^2$, from which it has been shown that the YM_2 on a sphere has a third-order phase transition at the critical area $A_c = \pi^2$. This is the well-known Douglas–Kazakov phase transition. It resembles the Gross–Witten–Wadia phase transition for the lattice two-dimensional multicolour gauge theory [10, 11].

The main characteristics of YM_2 are its invariance under area-preserving diffeomorphisms and the fact that there are no propagating degrees of freedom. These properties are not unique to YM_2, but rather are shared by a wide class of theories, called the generalized Yang–Mills theories (gYM_2s) [12, 13]. Various properties of gYM_2s have been studied, including their phase structure [14], the large-N behavior of the Wilson loops [15], and their double-scaling limit properties [16].

The phase structure of gYM_2s is an interesting issue from various points of view. The reason the Douglas–Kazakov (DK) phase transition occurs in YM_2 and gYM_2s is that in areas $A < A_c$ (the so-called weak regime) the dominant density function determined through a saddle-point analysis (ρ_w) is everywhere less than one. But for areas $A > A_c$ (the strong regime) ρ_w becomes greater than one in some points, which is not acceptable, and so ρ_w must be replaced by a new density (ρ_s). The transition from ρ_w to ρ_s in YM_2 induces a phase transition of order three. The DK phase transition has a richer structure in gYM_2s, and the order of which may be different from three, in so-called nontypical theories [14].

Another feature of the phase structure of gYM_2s, which does not exist in YM_2, is the possibility that ρ_w becomes negative somewhere. A negative density is not acceptable either, so it must be replaced by another density, which we denote by ρ_g (“g” for the “gapped phase”), in which the

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density function is zero in some region. In [17], a special gYM$_2$ has been studied in which the action is a combination of quartic and quadratic Casimir operators, and a third-order phase transition from the weak regime to the gapped regime has been observed. This observation is based on the classification established by Jurekiewicz and Zalewski [18].

In the present work, the structure of the transition between the weak regime and the gapped regime for arbitrary gYM$_2$s is studied. The technique used is similar to that used in [14, 15] to study the DK phase structure of gYM$_2$s. In this method, only the behavior of $\rho_{\rm w}$ near the critical point is needed. This is important, since the functional form of $\rho_{\rm w}$ is explicitly known for all gYM$_2$s, while this is not the case for $\rho_{\rm s}$ and $\rho_{\rm g}$. It is shown that the non-DK phase transitions between the weak and gapped regimes are in *almost* all gYM$_2$s of order three. There are, however, some special cases with fine-tuned couplings where one can have a phase transition of nontypical orders.

The scheme of the paper is the following. In Sect. 2, gYM$_2$s are briefly reviewed, and the method of calculating the order of non-DK phase transitions is discussed. In Sect. 3, this method is applied to the typical model discussed in [17], and it is proved that the order of this non-DK phase transition is three. Finally, in Sect. 4 a nontypical gYM$_2$ is introduced; the order of its weak-gapped transition is 5/2, and an outline is given of how to construct a general nontypical model with a non-DK phase transition of order $2+(1/k)$, where k is an integer greater than one.

2 The general method

The partition function of the generalized U(N) Yang–Mills theory on a sphere of area A is [12, 13, 19]

$$Z=\sum_r d_r^2\,{\rm e}^{-A\Lambda(r)}\,, \tag{1}$$

where r labels the irreducible representations of the group U(N), d_r is the dimension of the representation r, and

$$\Lambda(r)=\sum_{k=1}^{p}\frac{a_k}{N^{k-1}}C_k(r)\,. \tag{2}$$

C_k is the kth Casimir operator of the gauge group, and the a_ks are arbitrary constants. For the partition function (1) to be convergent, it is necessary that p in (2) be even and a_p be positive. The representations of U(N) are parameterized by the integers n_i, where $n_1\geq n_2\geq\cdots\geq n_N$.

In the large-N limit, it is convenient to introduce the continuous variable

$$\phi(x):=-n(x)-1+x\,, \tag{3}$$

where

$$0\leq x:=\frac{i}{N}\leq 1\,,\qquad n(x):=\frac{n_i}{N}\,. \tag{4}$$

The partition function (1) then becomes

$$Z=\int\prod_{0\leq x\leq 1}{\rm d}\phi(x){\rm e}^{S(\phi)}\,, \tag{5}$$

where

$$S(\phi):=N^2\left\{-A\int_0^1{\rm d}xG[\phi(x)]+\int_0^1{\rm d}x\int_0^1{\rm d}y\log|\phi(x)-\phi(y)|\right\} \tag{6}$$

(apart from an unimportant constant), and

$$G(\phi):=\sum_{k=1}^{p}(-1)^k a_k\phi^k\,. \tag{7}$$

Introducing the density function

$$\rho[\phi(x)]:=\frac{{\rm d}x}{{\rm d}\phi(x)}\,, \tag{8}$$

it is seen that it satisfies

$$\int_{-a}^{a}{\rm d}z\rho(z)=1\,, \tag{9}$$

where $[-a,a]$ is the interval corresponding to the values of ϕ. Here it is assumed that $G(\phi)$ is even, and therefore $\rho(z)$ is even as well. The condition $n_1\geq n_2\geq\cdots\geq n_N$ demands

$$0\leq\rho(z)\leq 1\,. \tag{10}$$

As N tends to infinity, the only representation that contributes to the partition function (5) is the so-called dominant (or classic) representation [20], satisfying

$$g(z)={\rm P}\int_{-a}^{a}{\rm d}z'\,\frac{\rho(z')}{z-z'}\,, \tag{11}$$

where P indicates the principal value of the integral, and

$$g(z):=\frac{A}{2}G'(z)\,. \tag{12}$$

The free energy of the theory is defined by

$$F:=-\frac{1}{N^2}\ln Z\,. \tag{13}$$

Using the standard method of solving the integral equation (11), the density function ρ is obtained, following [20], as

$$\rho(z)=\frac{\sqrt{a^2-z^2}}{\pi}\times\sum_{n,q=0}^{\infty}\frac{(2n-1)!!}{2^n n!(2n+2q+1)!}a^{2n}z^{2q}g^{(2n+2q+1)}(0)\,, \tag{14}$$

where the parameter a satisfies

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!(2n-1)!} a^{2n} g^{(2n-1)}(0) = 1 . \tag{15}$$

Here $g^{(n)}$ is the nth derivative of g.

The density function in (14), which we call ρ_{w}, should satisfy the conditions (10). It obviously depends on the area A and the parameters a_k, and there could be regions in the parameter space (of A and the a_ks) where these conditions are violated. In such regions, the density corresponding to the dominant representation is not ρ_{w}. The case where $\rho_{\mathrm{w}}(z)$ exceeds one for some z results in the well-known DK phase transition. Here we are interested in the case where $\rho_{\mathrm{w}}(z)$ becomes negative for some z. In this case ρ_{g}, the density corresponding to the dominant representation, is zero in some interval L_{g}, the length of which is defined $2b$. For the same parameters, ρ_{w} is negative in a region L'_{g}, the length of which is of the order of $2b$. Defining

$$\alpha := |\min(\rho_{\mathrm{w}})| , \tag{16}$$

one can use exactly the same arguments as used in [15] for the DK phase transition to show that

$$H_{\mathrm{g}}(z) - H_{\mathrm{w}}(z) \sim b^{2m+2} \sim \alpha^{1+(1/m)} \tag{17}$$

for large z. Here

$$H_{\mathrm{g,w}}(z) := \int \mathrm{d}y \frac{\rho_{\mathrm{g,w}}(y)}{z-y} , \tag{18}$$

$2m$ is the order of the first nonvanishing derivative of ρ_{w} at the point where ρ_{w} attains its minimum, and by large z is meant $|z| \gg a$.

Assume that the phase transition from ρ_{w} to ρ_{g} occurs at some critical value $A = A_{\mathrm{c}}$. If the order of the first nonvanishing derivative of α with respect to A at the point A_{c} is l, then (17) results in

$$H_{\mathrm{g}} - H_{\mathrm{w}} \sim (A - A_{\mathrm{c}})^{l[1+(1/m)]} . \tag{19}$$

Using (13) and the fact that the dominant representation maximizes S, it is seen that

$$\begin{aligned} F'(A) &= \int_0^1 \mathrm{d}x G[\phi(x)] , \\ &= \int \mathrm{d}y \rho(y) G(y) , \\ &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z H(z) G(z) , \end{aligned} \tag{20}$$

where the last integration is over a large contour, and in the last step (18) has been used. Thus one arrives at

$$F'_{\mathrm{g}}(A) - F'_{\mathrm{w}}(A) \sim (A - A_{\mathrm{c}})^{l[1+(1/m)]} , \tag{21}$$

and from this we have

$$F_{\mathrm{g}}(A) - F_{\mathrm{w}}(A) \sim (A - A_{\mathrm{c}})^{1+l[1+(1/m)]} . \tag{22}$$

This is our desired relation, and (22) shows that for typical theories where $l = m = 1$, the system exhibits a third-order phase transition at $A = A_{\mathrm{c}}$, but for special theories (with fine-tuned coupling constants) this order can in principle be different from three. The situation is completely analogous to that of the DK phase transition.

One can extend this argument to the case where several phase transitions occur, and each of these could correspond to a density exceeding one or becoming negative. Suppose that the density is already such that there are regions where it is identically one or zero, and suppose that varying the parameters creates a new region where the density either exceeds one or becomes negative. Let us call this density ρ_1, and the boundary which is violated (zero or one) B. This density should be corrected so that the corrected density ρ_2 does not cross the boundary B. For the density ρ_2, there is a new interval where ρ_2 is equal to B. Defining α as the maximum of $|\rho_1(z) - B|$, and $2b$ as the width of the region where the value of ρ_1 is not in $[0, 1]$, it is seen that exactly similar arguments lead to something like (22), where the left-hand side is $F_2(A) - F_1(A)$. So, even if there are several transitions of either kind (the density crossing zero or one), the order of each transition is similar to the case of the DK phase transition. Specially, any transition which is typical ($l = m = 1$) is a third-order transition.

3 A typical model

Consider the following typical model:

$$G(z) = c_2 z^2 + c_4 z^4 , \tag{23}$$

which has been investigated in [14, 17], where

$$c_4 > 0 . \tag{24}$$

Using (14) and (15), one finds

$$\rho_{\mathrm{w}}(z) = \frac{\mu}{\pi} \sqrt{a^2 - z^2} \left(z^2 + \frac{a^2}{2} + \beta \right) , \tag{25}$$

$$a^2 = -\frac{2\beta}{3} + \frac{2}{3} \sqrt{\frac{6}{\mu} + \beta^2} , \tag{26}$$

where

$$\begin{aligned} \beta &:= \frac{c_2}{2c_4} , \\ \mu &:= 2Ac_4 . \end{aligned} \tag{27}$$

There arise three distinct regions in the $(\beta - \mu)$ plane.

I $\beta > \sqrt{2/\mu}$.
Here $\rho_{\mathrm{w}}(z)$ has a maximum at $z = 0$.

II $-\sqrt{2/\mu} < \beta < \sqrt{2/\mu}$.
Here $\rho_{\mathrm{w}}(z)$ has a positive minimum at $z = 0$ and two maxima at $\pm z_0 := \pm\sqrt{(3a^2 - 2\beta)/6}$.

III $\beta < -\sqrt{2/\mu}$.
Here $\rho_{\mathrm{w}}(z)$ has a negative minimum at $z = 0$ and two maxima at $\pm z_0 := \pm\sqrt{(3a^2 - 2\beta)/6}$.

Clearly, in region III one must replace ρ_{w} with ρ_{g}, which is equal to zero in an even interval around $z=0$, and the system undergoes a non-DK phase transition from the weak regime to the gapped regime on the critical curve

$$\beta = -\sqrt{\frac{2}{\mu}} . \tag{28}$$

At this transition curve, the maximum value of $\rho_{\mathrm{w}}(z)$ is

$$\rho_{\mathrm{w}}(\pm z_0) = \left(\frac{2^{13}\mu}{3^6\pi^4}\right)^{1/4} . \tag{29}$$

One should also make sure that at this point ρ_{w} is still never greater than one, which sets a condition for β:

$$\beta \le \beta_0 := -\frac{128}{27\pi^2} . \tag{30}$$

Therefore, as μ increases, for $\beta<\beta_0$ the system goes from the weak regime to the gapped regime, while for $\beta>\beta_0$ the system goes from the weak regime to the strong regime, i.e. it undergoes the ordinary DK phase transition.

To obtain the order of the above non-DK phase transition, one must determine the parameters l and m in (22). Using (25), it is seen that

$$\rho''(0)|_{\mathrm{c}} = \left(\frac{128\mu^3}{\pi^4}\right)^{1/4} , \tag{31}$$

which is positive and this shows that $m=1$. To obtain l, one should investigate the behavior of α for fixed β with respect to μ. (Note that μ differs from A just by the multiplicative positive constant $2c_4$.) One has

$$\alpha = -\frac{\mu a}{\pi}\left(\frac{a^2}{2}+\beta\right) . \tag{32}$$

From (26), it is seen that the derivative of a with respect to μ at fixed β is a finite negative number. Noting that $(a^2+2\beta)$ vanishes at the transition, one obtains

$$\frac{\partial\alpha}{\partial\mu} = -\frac{\mu a^2}{\pi}\frac{\partial a}{\partial\mu} , \tag{33}$$

which is positive. So $l=1$, and the order of the transition is 3. This has been pointed out in [17], based on different arguments.

4 Nontypical models

Consider the following potential:

$$G_2(z) = \sum_{n=1}^{k+1} c_{2n} z^{2n} , \tag{34}$$

where c_{2k+2} is positive. Defining

$$\mu := (k+1)Ac_{2k+2} ,$$
$$\beta_n := \frac{k+1-n}{k+1}\frac{c_{2k+2-2n}}{c_{2k+2}} , \tag{35}$$

and using (14) and (15), one finds

$$\rho_{\mathrm{w}}(z) = \frac{\mu}{\pi}\sqrt{a^2-z^2}\sum_{q=0}^{k}\sum_{n=0}^{k-q}\gamma_n\beta_{k-n-q}z^{2q}$$
$$=: \frac{\mu}{\pi}\sqrt{a^2-z^2}\sum_{q=0}^{k}\delta_q z^{2q} , \tag{36}$$

and

$$\mu\sum_{n=1}^{k+1}\gamma_n\beta_{k+1-n} = 1 , \tag{37}$$

respectively, where

$$\gamma_n := \frac{(2n-1)!!}{2^n n!}a^{2n} . \tag{38}$$

The aim is to tune the coupling constants so that

$$\left.\sum_{q=0}^{k}\sum_{n=0}^{k-q}\gamma_n\beta_{k-n-q}z^{2q}\right|_{a=a_0} = z^{2k} . \tag{39}$$

This leads to

$$\left.\sum_{n=0}^{p}\gamma_n\beta_{p-n}\right|_{a=a_0} = \delta_{p,0} , \quad p\le k . \tag{40}$$

To solve this, one notices that

$$\sum_{n=0}^{\infty}\gamma_n s^n = (1-a^2 s)^{-1/2} . \tag{41}$$

It is seen that if one defines $\tilde\beta_n$ so that

$$\sum_{n=0}^{\infty}\tilde\beta_n s^n = \left(1-a_0^2 s\right)^{1/2} , \tag{42}$$

then

$$\left.\sum_{n=0}^{p}\gamma_n\tilde\beta_{p-n}\right|_{a=a_0} = \delta_{p,0} . \tag{43}$$

So β_n is the same as $\tilde\beta_n$, for $n\le k$:

$$\beta_n = -\frac{(2n-3)!!}{2^n n!}a_0^{2n} , \quad n\le k . \tag{44}$$

It is seen that β_0 is positive (in fact, one), while the other β_ns are negative. Now consider the coefficient of z^{2q} in the summation (36). One has

$$\delta_q = \gamma_{k-q}\left(1+\sum_{n=0}^{k-q-1}\frac{\gamma_n}{\gamma_{k-q}}\beta_{k-n-q}\right) . \tag{45}$$

It is seen that for $q<k$, the derivative of the parentheses with respect to a is positive. (Each term in the summation is a negative constant times a negative power of a.)

Knowing that it vanishes at $a = a_0$, one deduces that δ_q is positive for $a > a_0$, negative for $a < a_0$, and there is a value a_q less than a_0, so that the derivative of δ_q is positive for $a > a_q$. One then concludes that $\rho_{\rm w}$ is nonnegative for $a > a_0$, and $\rho_{\rm w}(0)$ is negative for $a < a_0$. Also, the derivative of $\rho_{\rm w}(0)$ with respect to a is positive for $a \geq a_0$. One can summarize this as follows:

$$\rho_{\rm w}(z) > 0\,, \quad a > a_0\,, \tag{46}$$
$$\rho_{\rm w}(0) < 0\,, \quad a < a_0\,, \tag{47}$$
$$\left.\frac{\partial \rho_{\rm w}(0)}{\partial a}\right|_{a\geq a_0} > 0\,. \tag{48}$$

Next, consider (37). The aim is to prove that with the choice (44), to every a larger than a_0 there corresponds a positive μ, and for $a \geq a_0$ the derivative of a with respect to μ is negative. One can rewrite (37) as

$$\begin{aligned}\frac{1}{\mu} &= -\tilde{\beta}_{k+1} + \sum_{n=0}^{k+1} \gamma_n \tilde{\beta}_{k+1-n}\,,\\ &= -\tilde{\beta}_{k+1} + \delta_{-1}\,.\end{aligned} \tag{49}$$

Therefore, exactly repeating the arguments used to arrive at (46) to (48), one concludes that for $a \geq a_0$ the right-hand side of (49) is positive and the derivative of the right-hand side of (49) with respect to a is positive:

$$\mu(a) > 0\,, \quad a \geq a_0, \tag{50}$$
$$\frac{\partial \mu}{\partial a} < 0\,, \quad a \geq a_0\,. \tag{51}$$

The final picture is as follows. Increasing μ from zero, a decreases from infinity, so that at $\mu = \mu_0$, one arrives at $a = a_0$:

$$\mu_0 := \frac{2^{k+1}(k+1)!}{(2k-1)!!\,a_0^{2k+2}}\,. \tag{52}$$

For $\mu < \mu_0$, the density is nonnegative, while as μ exceeds μ_0, the density becomes negative at $z = 0$. Equations (48) and (51) show that the derivative of α with respect to μ is nonvanishing at $\mu = \mu_0$. Hence,

$$l = 1\,. \tag{53}$$

To find the value of m, one notes that at the transition point the first nonvanishing derivative of $\rho_{\rm w}$ at the origin is the $(2k)$th derivative. So we have

$$m = k\,. \tag{54}$$

From these equations, one finds that the order of the transition is $[2 + (1/k)]$.

There remains one point to take into account, which is to make sure that for $\mu \leq \mu_0$ the density does not exceed one. To address this, one notices that the transformation

$$\begin{aligned}&a \to \sigma a\,,\\ &a_0 \to \sigma a_0\,,\\ &\mu \to \sigma^{-2k-2}\mu\,,\\ &z \to \sigma z\,,\\ &\rho \to \sigma^{-1}\rho\,,\end{aligned} \tag{55}$$

where σ is an arbitrary positive constant, leaves all of the relations intact. So, using a sufficiently large σ ensures that the DK transition does not occur before the transition from the weak regime to the gapped regime.

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